N63-19371

NASA TN D-1898

NASA TN D-1898

TECHNICAL NOTE

D-1898

COMPUTATION OF GENERAL PLANETARY PERTURBATIONS, PART I

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON August 1963

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SUMMARY

A computer program for automatic computation of first order general planetary perturbations is described. The program is based on Hansen's theory as given in the Auseinandersetzung. As examples the general perturbations of six minor planets are given.

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INTRODUCTION

The need for general perturbations of minor planets, comets and artificial satellites has been widely recognized, and these series developments remain of interest despite well developed techniques for numerical integration on electronic computers. For purposes of long range predictions and studies of stability of orbits with peculiar elements, the perturbations in series are of particular importance. As artificial satellites are launched into orbits with larger semi-major axes, the study of their behavior under the influence of the moon will bear close resemblance to the planetary or cometary problem. Advances in the speed and capabilities of electronic computers have reduced the programming of general perturbations to a reasonable task.

With these facts in mind a program was developed for automatic machine computation of general planetary perturbations using Hansen's theory as given in the Auseinandersetzung (1857) (Reference 1). Hill's modification for computing the perturbations in the radius vector (Reference 2) and Herrick's suggested use of Gibbs' vectorial elements have been included. Hansen's method was chosen because it can be applied to a wide range of eccentricities and inclinations. No exposition of the theory will be given since no modifications have been made. The details involved in the computational procedure are presented in Appendix A. Perturbations for several minor planets are given in Appendix B.

GENERAL DISCUSSION OF COMPUTATIONS

The computational procedure was taken from Herget (Reference 3) except that the Laplace coefficients are computed directly. The program takes the elements of the disturbed and disturbing bodies as the input data and prints out the coefficients in the series for the perturbations of first order as the results. The time required on the IBM 7094 for each set of perturbations is approximately 1 minute per planet.

The following set of equations is used in developing the series for the perturbations (Reference 3, Equations 8, 6):*

$$3\,a\,\Omega = 3\,m'\,a\,\Delta^{-1} + (-3\mathcal{H}) ,$$

$$a\,r\,\frac{\partial\Omega}{\partial r} = m'\,a\,\Delta^{-3}\left(\frac{r'^2 - r^2}{2}\right) - \frac{1}{6}\,(3\,a\,\Omega) + \frac{1}{2}\,(-3\mathcal{H}) ,$$

$$a^2\,\frac{\partial\Omega}{\partial Z} = m'\,a\left(\Delta^{-3} - r'^{-3}\right) Z' ,$$

$$W = \int\left[\frac{1}{3}\,M\,\frac{\partial(3\,a\,\Omega)}{\partial E} + N\,a\,r\,\frac{\partial\Omega}{\partial r}\right] dE ,$$

$$R = \int Q\,a^2\,\frac{\partial\Omega}{\partial Z}\,dE ,$$

$$n\delta z = \int \overline{W}\,(1 - e\cos E)\,dE ,$$

$$\nu = -\frac{1}{6}\,X_0 - \frac{e}{6}\,X_1 - \frac{1}{2}\,\overline{W} ,$$

$$u = \overline{R} .$$

A major part of the computation is spent in the development of Δ^{-1} and Δ^{-3} where Δ is the distance between the disturbed and disturbing bodies. These quantities are expanded into double Fourier series in terms of the eccentric anomalies by taking the first sixteen Laplace coefficients and applying harmonic analysis with twenty-four equally spaced values of the eccentric anomaly of the disturbed body. Representing the argument of any term in the form $(iE - jE')$, the terms which are computed correspond to $j = 0$ through 15 and $i = j - 11$ through $j + 11$. The derivatives of the disturbing function

$$\frac{\partial(3\,a\,\Omega)}{\partial E} , \quad a\,r\,\frac{\partial\Omega}{\partial r} , \quad \text{and} \quad a^2\,\frac{\partial\Omega}{\partial Z}$$

are obtained by simple operations on the series Δ^{-1} and Δ^{-3}. These series are then transformed to arguments of the form $(iE - j\phi)$ where

$$\phi = \frac{n'}{n}\left(E - g_0\right) + g_0'$$

*See Appendix A for notation used and computational details.

by applying the Bessel transformations. The multiplications by the M, N, and Q expressions which are given in the Collection of Formulas are combined with the term by term integrations by forming sums of products of coefficients and dividing by $i - j\, n'/n$. The constant terms yield terms factored by E after the integration, and these are converted to coefficients of time by replacing E by $nt + e \sin E$. The replacement bar operation is accomplished by considering the temporary angle H to be the same as E and combining corresponding coefficients. After the formal integration is completed, the constants of integration are determined so as to satisfy the initial conditions.

The perturbations are used in the following manner. For any given time, solve the Kepler's equation

$$E - e \sin E = g_0 + n\left(t - t_0\right)$$

to obtain the undisturbed eccentric anomaly E. E must not be reduced modulus 360° because it appears in the non-integer multiples through ϕ. With this value of E the argument for each term is computed:

$$iE - j\phi = \left(i - j\,\frac{n'}{n}\right)E + j\left(\frac{n'}{n}\, g_0 - g_0'\right) .$$

Evaluate the series for $n\delta z$, ν and u by multiplying the coefficients by the cosine or sine of $iE - j\phi$ and adding the terms of the series. With the value of $n\delta z$ Kepler's equation is solved for the disturbed eccentric anomaly

$$\bar{E} - e \sin \bar{E} = g_0 + n\left(t - t_0\right) + n\delta z .$$

The disturbed position vector is then given by

$$\mathbf{r} = (1 + \nu)\left[\mathbf{A}(\cos \bar{E} - e) + \mathbf{B} \sin \bar{E} + \mathbf{C}u\right] .$$

The velocity vector may also be determined, but with less accuracy, by evaluating the derivatives of the series for the perturbations. Thus for any given time the osculating elements of the disturbed motion may be obtained. Taking $\mathbf{v} = d\mathbf{r}/d\tau$ where $\tau = k\left(t - t_0\right)$ and k is the Gaussian constant, we have

$$\mathbf{v} = \left[\mathbf{A}(\cos \bar{E} - e) + \mathbf{B} \sin \bar{E} + \mathbf{C}u\right]\frac{d\nu}{d\tau}$$

$$+ (1 + \nu)\left\{\left[-\mathbf{A} \sin \bar{E} + \mathbf{B} \cos \bar{E}\right]\frac{d\bar{E}}{d\tau} + \mathbf{C}\,\frac{du}{d\tau}\right\},$$

where

$$\frac{d\bar{E}}{d\tau} = \frac{a^{-3/2} + \frac{d}{d\tau}(n\delta z)}{1 - e \cos \bar{E}} ;$$

and the derivatives of the perturbations are computed by using the relation

$$\frac{d}{d\tau} = \frac{1}{r\sqrt{a}} \frac{d}{dF} .$$

The example given by Herget (Reference 3) of the perturbations of (1286) Banachiewizca by Jupiter was used to check all the intermediate results during the programming process. After the program was completed, a comparison was made with the first order perturbations of (13) Egeria computed by Hansen and given in his original work (Reference 1). The agreement in the perturbations due to Jupiter and Saturn in this case were most encouraging, the largest difference between corresponding periodic terms was less than 1 second of arc, while for the secular and mixed terms the differences were of the order of 10^{-4} seconds of arc or less. For the small perturbations due to Mars there was some disagreement. These comparisons were made without including the constants of integration, since Hansen computed these terms once including the perturbations due to Jupiter, Saturn and Mars together. Additional comparisons were made with various perturbations which have been computed using the same method. The elements given by the original authors were used for these comparisons. The examples of perturbations given in Appendix B are based on the elements of minor planets given in Reference 4 and the elements of Jupiter given by Clemence (Reference 5). The main differences in the perturbations computed using different sets of elements appear in the terms affected by the constants of integration and in the long period terms associated with small divisors.

CONCLUSION

We now have the facility for automatic computation of Hansen's first order planetary perturbations. These perturbations are sufficiently accurate for the practical purposes of identification and producing ephemerides in the case of planetary-type motion.

Several authors have made contributions to the development into Fourier series of lunar perturbations of artificial satellites. The works of Kozai (Reference 6); Musen, Bailie and Upton (Reference 7); and Kaula (Reference 8) should be mentioned. The analytic development in powers of the ratio of the semi-major axes converges rapidly for close earth satellites, but for more distant satellites the convergence is slow. For the latter cases one must apply harmonic analysis as in Hansen's planetary theory. For this reason the program described in this article is now the most efficient method of treating periodic perturbations of cislunar satellites.

The continuation of this work has several different aspects. A problem of considerable interest and importance is increasing the range of applicability so that general perturbations of highly eccentric orbits can be computed. A modification of the present program which accomplishes this purpose by direct double harmonic analysis will be described in Part II. A corresponding program is planned using the mean anomaly as the independent variable for greater convenience in evaluating the perturbations. For greater accuracy it is desirable to have a program for the computation of perturbations of higher order. For this purpose the equations in rectangular coordinates given by Musen and Carpenter (Reference 9) have a convenient form for programming. First order perturbations in rectangular coordinates will be compared with Hansen's perturbations. It also would be desirable to have a program using mean elements and a program for orbit correction based on general perturbations.

ACKNOWLEDGMENTS

The author expresses his gratitude to Dr. Peter Musen, who suggested the work on general planetary perturbations and gave generously of his time in many helpful discussions. Thanks are also given to Neal Paris, who assisted in modifying the program to include direct double harmonic analysis.

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Appendix A

Collection of Formulas

NOTATION

g_0, a, e, n, ω, Ω, i,	the osculating elements of the disturbed body
g_0', a', e', n', ω', Ω', i'	the orbital elements of the disturbing body
Ω	the disturbing function
$\mathbf{r}$	the position vector of the disturbed body
$\mathbf{v}$	the velocity vector of the disturbed body
$r = \lvert\mathbf{r}\rvert$	
$\mathbf{r}'$	the position vector of the disturbing body
$r' = \lvert\mathbf{r}\rvert$	
Δ	the mutual distance $\Delta^2 = r^2 + r'^2 - 2\,\mathbf{r}\cdot\mathbf{r}'$
m'	the mass of the disturbing body
$\mathcal{H} = m'\,a\mathbf{r}\cdot\mathbf{r}'/r'^3$	
H	the fictitious eccentric anomaly to be replaced by E after the integration
E	the eccentric anomaly of the disturbed body
$\mathbf{A}$	the vector of length $\underline{a}$ in the direction of perigee of the undisturbed orbit
$\mathbf{B}$	the vector of length $a\sqrt{1-e^2}$ in the direction 90° in advance of perigee of the undisturbed orbit
$\mathbf{C}$	the vector of length $\underline{a}$ in the direction of the angular momenturn of the undisturbed orbit; the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are referred to the equatorial system of coordinates
Z	the coordinate in the direction of $\mathbf{C}$
$n\delta z$	the perturbation in the mean anomaly
ν	the perturbation in the length of the radius vector
u	the perturbation normal to the orbit plane
$\phi = \frac{n'}{n}(E - g_0) + g_0'$	
E_0	the eccentric anomaly of the disturbed body at the epoch
ϵ	the obliquity of the ecliptic
$b_s^{(j)}$	the Laplace coefficients

For purposes of computation, the remaining symbols used in the Collection of Formulas are defined by the expressions in which they appear.

THE PROCEDURE

1. Take the osculating elements

$$a,\ e,\ i,\ \omega,\ \Omega,\ g_0,\ n$$

of the disturbed planet and the elements

$$a',\ e',\ i',\ \omega',\ \Omega',\ g_0',\ n',\ m'$$

of the disturbing planet as input data.

2. Evaluate the vectors

$$\mathbf{A} = a\,G(\epsilon)\cdot\begin{pmatrix}\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i \\ \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i \\ \sin\omega\sin i\end{pmatrix},$$

$$\mathbf{B} = a\sqrt{1-e^2}\,G(\epsilon)\cdot\begin{pmatrix}-\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i \\ -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i \\ \cos\omega\sin i\end{pmatrix},$$

$$\mathbf{C} = a\,G(\epsilon)\cdot\begin{pmatrix}\sin\Omega\sin i \\ -\cos\Omega\sin i \\ \cos i\end{pmatrix},$$

and the corresponding primed vectors for the disturbing planet. The rotation matrix $G(\epsilon)$ refers the vectors to the equatorial system of coordinates:

$$G(\epsilon) = \begin{pmatrix}1 & 0 & 0 \\ 0 & \cos\epsilon & -\sin\epsilon \\ 0 & \sin\epsilon & \cos\epsilon\end{pmatrix}.$$

3. Compute the dot products

$$\mathbf{A}\cdot\mathbf{A}',\ \mathbf{A}\cdot\mathbf{B}',\ \mathbf{B}\cdot\mathbf{A}',\ \mathbf{B}\cdot\mathbf{B}',\ \mathbf{C}\cdot\mathbf{A}',\ \mathbf{C}\cdot\mathbf{B}'.$$

4. Carry out steps 4a through 4e for each of the following values of the eccentric anomaly E of the disturbed planet: $E = 0°, 15°, 30°, \cdots, 345°$

a. Compute the following quantities:

$$r = a(1 - e\cos E)$$

$$K\cos\psi = 2e'a'^2 - 2e\mathbf{A}\cdot\mathbf{A}' + 2\mathbf{A}\cdot\mathbf{A}'\cos E + 2\mathbf{B}\cdot\mathbf{A}'\sin E$$

$$K\sin\psi = -2e\mathbf{A}\cdot\mathbf{B}' + 2\mathbf{A}\cdot\mathbf{B}'\cos E + 2\mathbf{B}\cdot\mathbf{B}'\sin E$$

$$H = a'^2\left(1 - 2e'^2\right) + r^2 + e'(K\cos\psi) \; .$$

b. Solve by iteration for C, q, and Q starting with $Cw/q^2 = 0$, where $w = a'^2 e'^2$:

$$q\cos Q = \frac{K\cos\psi}{1 + Cw/q^2}$$

$$q\sin Q = \frac{K\sin\psi}{1 - Cw/q^2}$$

$$C = H + \frac{w}{q^2}(q\sin Q)^2 \; .$$

c. Compute the following quantities:

$$A = \frac{q}{C + \sqrt{C^2 - q^2}}$$

$$Q' = Q - E$$

$$1.5\,\frac{w}{q}\cos Q'$$

$$1.5\,\frac{w}{q}\sin Q'$$

$$\frac{15}{16}\left(\frac{w}{q}\right)^2$$

$$\frac{15}{16}\left(\frac{w}{q}\right)^2\cos 2Q'$$

$$\frac{15}{16}\left(\frac{w}{q}\right)^2\sin 2Q'$$

d. Compute the Laplace coefficients $b_s^{(j)}$ for $s = 1/2$ and $3/2$ and $j = 0$ through 15 as follows:

i. Compute $b_{1/2}^{(14)}$ and $b_{1/2}^{(15)}$ by numerical evaluation of the integral

$$b_{1/2}^{(j)} = \frac{4}{\pi} A^j \int_0^{\pi/2} \frac{\sin^{2j}\theta}{\sqrt{1 - A^2 \sin^2\theta}}\, d\theta \; .$$

ii. Compute $b_{1/2}^{(j)}$ for $j = 13, 12, \cdots, 0$ by the recurrence relation

$$b_{1/2}^{(j)} = \frac{2j+2}{2j+1}\left(A + \frac{1}{A}\right) b_{1/2}^{(j+1)} - \frac{2j+3}{2j+1}\, b_{1/2}^{(j+2)} \; .$$

iii. Compute $b_{3/2}^{(15)}$ from the equation

$$b_{3/2}^{(15)} = -29\, \frac{1 + A^2}{\left(1 - A^2\right)^2}\, b_{1/2}^{(15)} + 58\, \frac{A}{\left(1 - A^2\right)^2}\, b_{1/2}^{(14)} \; .$$

iv. Compute $b_{3/2}^{(j)}$ for $j = 0, 1, \cdots, 14$ from the recurrence relation

$$b_{3/2}^{(j)} \quad (2j+1)\, \frac{\left(1 + A^2\right)}{\left(1 - A^2\right)^2}\, b_{1/2}^{(j)} - 2(2j+1)\, \frac{A}{\left(1 - A^2\right)^2}\, b_{1/2}^{(j+1)} \; .$$

e. By taking $k = q/2A$, compute the following for $s = 1/2$ and $3/2$ and $j = 0$ through 15

$$k^{-s}\, b_s^{(j)} \cos j\, Q'$$

$$k^{-s}\, b_s^{(j)} \sin j\, Q' \; .$$

5. Compute the coefficients $C_{j,h}^{*}$, $S_{j,h}^{*}$, $C_{j,h}^{*\prime}$, and $S_{j,h}^{*\prime}$ in the following expansions by harmonic analysis of the tabulated values:

$$k^{-s}\, b_s^{(j)} \cos j\, Q' = \sum_h \left[C_{j,h}^{*} \cos(hE) + S_{j,h}^{*} \sin(hE)\right]$$

$$k^{-s}\, b_s^{(j)} \sin j\, Q' = \sum_h \left[C_{j,h}^{*\prime} \cos(hE) + S_{j,h}^{*\prime} \sin(hE)\right] ,$$

for $s = 1/2$ and $3/2$; $j = 0$ through 15 and $h = 0$ through 11.

6. Compute the coefficients c_{ij} and s_{ij} in the following expansions

$$\left[C - q\cos(E - E' + Q')\right]^{-s} = \sum_i \sum_j \left[c_{ij} \cos(iE - jE') + s_{ij} \sin(iE - jE')\right]$$

for $s = 1/2$ and $3/2$; $j = 0$ through 15 and $i = j - 11$ through $j + 11$ using the formula

$$[C - q\cos(E - E' + Q')]^{-s} = \sum_j \left[\left(k^{-s} b_s^{(j)} \cos jQ'\right) \cos j(E - E')\right.$$

$$\left. - \left(k^{-s} b_s^{(j)} \sin jQ'\right) \sin j(E - E')\right] .$$

7. Evaluate the coefficients C_{ij} and S_{ij} in the series

$$\left[1 - \frac{w}{q}\cos(E - E' + Q')\right]^{-s} = 1 + \sum_{i,j} \left[C_{i,j}\cos(iE - jE') + S_{i,j}\sin(iE - jE')\right]$$

for $s = 1/2$ and $3/2$, and $j = 0$, 1, and 2 by harmonic analysis over E of the coefficients in the first three terms of the binomial expansion of the left-hand side.

8. Obtain double Fourier series for Δ^{-1} and Δ^{-3} from the product

$$\Delta^{-2s} = [C - q\cos(E - E' + Q')]^{-s} \cdot \left[1 - \frac{w}{q}\cos(E + E' + Q')\right]^{-s} .$$

9. Expand $(a'/r')^3$ in a cosine series in E' by harmonic analysis of the expression

$$\left(\frac{a'}{r'}\right)^3 = \frac{1}{(1 - e'\cos E')^3} .$$

10. Evaluate the coefficients in the expression

$$\begin{aligned}(2\mathbf{r}\cdot\mathbf{r}') = {} & 2ee'\,\mathbf{A}\cdot\mathbf{A}' \\ & - 2e'\,\mathbf{A}\cdot\mathbf{A}'\cos E - 2e'\,\mathbf{B}\cdot\mathbf{A}'\sin E \\ & + [\mathbf{A}\cdot\mathbf{A}' - \mathbf{B}\cdot\mathbf{B}']\cos(-E - E') - [\mathbf{B}\cdot\mathbf{A}' + \mathbf{A}\cdot\mathbf{B}']\sin(-E - E') \\ & - 2e\,\mathbf{A}\cdot\mathbf{A}'\cos(-E') + 2e\,\mathbf{A}\cdot\mathbf{B}'\sin(-E') \\ & + [\mathbf{A}\cdot\mathbf{A}' + \mathbf{B}\cdot\mathbf{B}']\cos(E - E') + [\mathbf{B}\cdot\mathbf{A}' - \mathbf{A}\cdot\mathbf{B}']\sin(E - E') .\end{aligned}$$

11. Evaluate the coefficients in the series

$$-3\mathcal{H} = -\frac{3m'a}{2a'^3}\left(\frac{a'}{r'}\right)^3 (2\mathbf{r}\cdot\mathbf{r}') .$$

12. Evaluate the coefficients in the series

$$3a\Omega = 3m' a\Delta^{-1} + (-3\mathcal{H}) \; .$$

13. Evaluate the coefficients in the series $\partial(3a\Omega)/\partial E$ from those of $3a\Omega$ by differentiating term by term.

14. Evaluate the coefficients in the expression

$$\left(r'^2 - r^2\right) = a'^2 - a^2 + \frac{a'^2 e'^2}{2} - \frac{a^2 e^2}{2}$$

$$- 2a'^2 e' \cos(-E') + 2a^2 e \cos E$$

$$+ \frac{a'^2 e'^2}{2} \cos(-2E') - \frac{a^2 e^2}{2} \cos 2E \; .$$

15. Evaluate the coefficients in the series

$$ar \frac{\partial \Omega}{\partial r} = \frac{1}{2} m' a \left(r'^2 - r^2\right) \Delta^{-3} - \frac{1}{6}(3a\Omega) + \frac{1}{2}(-3\mathcal{H}) \; .$$

16. Evaluate the coefficients in the expression

$$Z' = -\mathbf{C} \cdot \mathbf{A}' e' + \mathbf{C} \cdot \mathbf{A}' \cos(-E') - \mathbf{C} \cdot \mathbf{B}' \sin(-E') \; .$$

17. Evaluate the coefficients in the series

$$a^2 \frac{\partial \Omega}{\partial Z} = m' a \left(\Delta^{-3} - r'^{-3}\right) Z' \; .$$

18. Compute the Bessel function coefficients:

$$P_k^{(j)} = \frac{j}{k} J_{k-j}(ke') \quad \text{for} \quad k, j = 1, 2, 3, \cdots, 15,$$

$$P_0^{(1)} = -\frac{e'}{2} ,$$

$$P_0^{(j)} = 0 \quad \text{for} \quad j \neq 1 ,$$

where

$$J_m(x) = \sum_{l=0} \frac{(-1)^l \left(\frac{x}{2}\right)^{2l+m}}{l!\,(l+m)!}$$

and

$$J_{-m}(x) = (-1)^m J_m(x) .$$

19. Transform the series for $\partial(3a\Omega)/\partial E$, $ar(\partial\Omega/\partial r)$, and $a^2(\partial\Omega/\partial Z)$ from arguments of the form $iE - jE'$ to arguments of the form $iE - jg'$ using the expansion

$$\begin{matrix}\cos\\ \sin\end{matrix}(iE - jE') = \sum_{k=-\infty}^{+\infty} P_k^{(j)} \begin{matrix}\cos\\ \sin\end{matrix}(iE - kg')$$

for each term.

20. Transform the series for $\partial(3a\Omega)/\partial E$, $ar(\partial\Omega/\partial r)$, and $a^2(\partial\Omega/\partial Z)$ from arguments $iE - jg'$ to arguments $iE - j\phi$, where $\phi = (n'/n)E - (n'/n)g_0 + g_0'$, using the expansion

$$\begin{matrix}\cos\\ \sin\end{matrix}(iE - jg') = \sum_{k=-\infty}^{+\infty} J_{k-i}\left(je\,\frac{n'}{n}\right)\begin{matrix}\cos\\ \sin\end{matrix}(kE - j\phi) ,$$

where the Bessel functions are computed as in step 18.

21. Evaluate the coefficients in the expression

$$Q = e\sin(E) - \frac{1}{2}e^2\sin(2E) + \frac{1}{2}e^2\sin(H+E)$$

$$-\frac{3}{2}e\sin(H) + \left(1 + \frac{1}{2}e^2\right)\sin(H-E) - \frac{1}{2}e\sin(H-2E) .$$

22. Evaluate the coefficients in the series

$$R = \int Q\left[a^2\frac{\partial\Omega}{\partial Z}\right]dE$$

(constants of integration are determined later). If, from step 21, we write Q in the form

$$Q = \sum_{k,l} Q_{k,l}\sin(kE + lH)$$

and if $a^2(\partial\Omega/\partial Z)$ is written as

$$a^2\frac{\partial\Omega}{\partial Z} = \sum_{i,j}\left[C_{i,j}\cos(iE - j\phi) + S_{i,j}\sin(iE - j\phi)\right] ,$$

then R may be written as

$$R = c_0 E + c_1 E \cos H + s_1 E \sin H$$

$$+ \sum_{i,j} \Big[A_{-1,i,j} \cos(iE - j\phi - H) + B_{-1,i,j} \sin(iE - j\phi - H)$$

$$+ A_{0,i,j} \cos(iE - j\phi) + B_{0,i,j} \sin(iE - j\phi)$$

$$+ A_{+1,i,j} \cos(iE - j\phi + H) + B_{+1,i,j} \sin(iE - j\phi + H) \Big] \;;$$

where

$$c_0 = \frac{1}{2} \sum_k \left(S_{k,0} - S_{-k,0}\right) Q_{k,0} \;,$$

$$c_1 = \frac{1}{2} \sum_k \left(S_{k,0} - S_{-k,0}\right) Q_{k,1} \;,$$

$$s_1 = \frac{1}{2} \sum_k \left(C_{k,0} + C_{-k,0}\right) Q_{k,1} \;,$$

$$A_{-1,i,j} = \frac{1}{2\left(i - j \frac{n'}{n}\right)} \sum_k C_{i+k,j} \; Q_{k,1} \;,$$

$$B_{-1,i,j} = \frac{1}{2\left(i - j \frac{n'}{n}\right)} \sum_k S_{i+k,j} \; Q_{k,1} \;,$$

$$A_{0,i,j} = \frac{1}{2\left(i - j \frac{n'}{n}\right)} \sum_k \left(C_{i+k,j} - C_{i-k,j}\right) Q_{k,0} \;,$$

$$B_{0,i,j} = \frac{1}{2\left(i - j \frac{n'}{n}\right)} \sum_k \left(S_{i+k,j} - S_{i-k,j}\right) Q_{k,0} \;,$$

$$A_{+1,i,j} = \frac{1}{2\left(i - j \frac{n'}{n}\right)} \sum_k - C_{i-k,j} \; Q_{k,1} \;,$$

$$B_{+1,i,j} = \frac{1}{2\left(i - j \frac{n'}{n}\right)} \sum_k - S_{i-k,j} \; Q_{k,1}$$

except that for $i = j = 0$ all these coefficients are zero. For $j = 0$ the coefficients with negative values of i are combined with the coefficients with positive values of i. That is, $A_{-1,-i,0}$ is added to $A_{+1,i,0}$ etc., where i is positive.

23. Evaluate the coefficients in the series

$$u = \overline{R} \; ;$$

The bar operator means that u is obtained from R by considering the argument H to be the same as E and adding corresponding coefficients. Thus, u will contain

$$c_0 E + c_1 E \cos E + s_1 E \sin E$$

plus pure periodic terms where, for example, the coefficient of $\cos(iE - j\phi)$ will be

$$A_{-1,i+1,j} + A_{0,i,j} + A_{+1,i-1,j} \; .$$

24. Evaluate the coefficients in the expressions

$$\frac{M}{3} = -\frac{1 - \frac{1}{2}e^2}{\left(1 - e^2\right)} + \frac{2e}{3\left(1 - e^2\right)} \cos(E) - \frac{e^2}{6\left(1 - e^2\right)} \cos(2E)$$

$$- \frac{e}{\left(1 - e^2\right)} \cos(H) + \frac{e^2}{3\left(1 - e^2\right)} \cos(H + E) + \frac{4 - e^2}{3(1 - e^2)} \cos(H - E)$$

$$- \frac{e}{3\left(1 - e^2\right)} \cos(H - 2E)$$

and

$$N = \frac{e}{\left(1 - e^2\right)} \sin(E) - \frac{e^2}{2\left(1 - e^2\right)} \sin(2E) + \frac{e^2}{\left(1 - e^2\right)} \sin(H + E)$$

$$- \frac{e}{\left(1 - e^2\right)} \sin(H) - \frac{2 - e^2}{\left(1 - e^2\right)} \sin(H - E) + \frac{e}{\left(1 - e^2\right)} \sin(H - 2E) \; .$$

25. Evaluate the coefficients in the series

$$W = \int \left[\frac{M}{3} \cdot \frac{\partial(3a\Omega)}{\partial E} + N \cdot ar \frac{\partial \Omega}{\partial r}\right] dE \; .$$

If

$$\frac{M}{3} = \sum_{k,\, l} M_{k,l} \cos(kE + lH) \ ,$$

$$N = \sum_{k,\, l} N_{k,l} \sin(kE + lH) \ ,$$

$$\frac{\partial(3a\Omega)}{\partial E} = \sum_{i,\, j} \left[C_{i,j} \cos(iE - j\phi) + S_{i,j} \sin(iE - j\phi)\right] \ ,$$

and

$$ar \frac{\partial \Omega}{\partial r} = \sum_{i,\, j} \left[c_{i,j} \cos(iE - j\phi) + s_{i,j} \sin(iE - j\phi)\right] \ ,$$

then W may be written in the same form as R in step 22 with

$$c_0 = \frac{1}{2} \sum_k \left[\left(C_{k,0} + C_{-k,0}\right) M_{k,0} + \left(s_{k,0} - s_{-k,0}\right) N_{k,0}\right] \ ,$$

$$c_1 = \frac{1}{2} \sum_k \left[\left(C_{k,0} + C_{-k,0}\right) M_{k,1} + \left(s_{k,0} - s_{-k,0}\right) N_{k,1}\right] \ ,$$

$$s_1 = \frac{1}{2} \sum_k \left[\left(-S_{k,0} + S_{-k,0}\right) M_{k,1} + \left(c_{k,0} + c_{-k,0}\right) N_{k,1}\right] \ ,$$

$$A_{-1,i,j} = \frac{1}{2\left(i - j\,\frac{n'}{n}\right)} \sum_k \left(-S_{i+k,j} M_{k,1} + c_{i+k,j} N_{k,1}\right) \ ,$$

$$B_{-1,i,j} = \frac{1}{2\left(i - j\,\frac{n'}{n}\right)} \sum_k \left(C_{i+k,j} M_{k,1} + s_{i+k,j} N_{k,1}\right) \ ,$$

$$A_{0,i,j} = \frac{1}{2\left(i - j\,\frac{n'}{n}\right)} \sum_k \left[\left(-S_{i+k,j} - S_{i-k,j}\right) M_{k,0} + \left(c_{i+k,j} - c_{i-k,j}\right) N_{k,0}\right] \ ,$$

$$B_{0,i,j} = \frac{1}{2\left(i - j\,\frac{n'}{n}\right)} \sum_k \left[\left(C_{i+k,j} + C_{i-k,j}\right) M_{k,0} + \left(s_{i+k,j} - s_{i-k,j}\right) N_{k,0}\right] \ ,$$

$$A_{+1,i,j} = \frac{1}{2\left(i - j\frac{n'}{n}\right)} \sum_k \left(-S_{i-k,j} M_{k,1} - C_{i-k,j} N_{k,1}\right) ,$$

$$B_{+1,i,j} = \frac{1}{2\left(i - j\frac{n'}{n}\right)} \sum_k \left(C_{i-k,j} M_{k,1} - S_{i-k,j} N_{k,1}\right) .$$

26. Evaluate the coefficients in $\bar{W}$ from W by considering H to be the same as E and combining coefficients as in step 23.

27. Evaluate the coefficients in the series

$$\nu = -\frac{\left(X_0 + eX_1\right)}{6} - \frac{\bar{W}}{2} ,$$

where X_0 is that part of the W series which does not contain H in the argument and X_1 is the remaining part of the W series with the temporary argument H set equal to zero. Thus, for example, the coefficient of $\cos(iE - j\phi)$ in ν would be

$$-\frac{1}{6}\left[A_{0,i,j} + e\left(A_{-1,i,j} + A_{+1,i,j}\right)\right] - \frac{1}{2}\left[A_{-1,i+1,j} + A_{0,i,j} + A_{+1,i-1,j}\right] .$$

28. Evaluate the coefficients in the series

$$n\delta z = \int \bar{W}(1 - e\cos E)\, dE .$$

The coefficient of E^2 in $n\delta z$ should be zero.

29. Evaluate the coefficients in the series for $d\nu/dE$ and du/dE from the corresponding series for ν and u by differentiating term by term.

30. Convert the terms factored by E in each series to terms factored by time by replacing E by $nt + e\sin E$.

31. Compute numerical values of $n\delta z$, ν, u, $\bar{W}$, $d\nu/dE$, and du/dE corresponding to the epoch of osculation by evaluating the series with E set equal to E_0. Denote these values by zero subscripts.

32. Evaluate the constants of integration:

$$l_1 = -u_0 \frac{\cos E_0}{1 - e\cos E_0} + \left(\frac{du}{dE}\right)_0 \frac{\sin E_0}{1 - e\cos E_0}$$

$$l_2 = -u_0 \frac{\sin E_0}{1 - e\cos E_0} - \left(\frac{du}{dE}\right)_0 \frac{\cos E_0 - e}{1 - e\cos E_0}$$

$$k_1 = -\left[4\overline{W}_0 + 6\nu_0\right] \frac{\cos E_0}{1 - e\cos E_0} - 2\left(\frac{d\nu}{dE}\right)_0 \frac{\sin E_0}{1 - e\cos E_0}$$

$$k_2 = -\left[4\overline{W} + 6\nu_0\right] \frac{\sin E_0}{1 - e\cos E_0} + 2\left(\frac{d\nu}{dE}\right)_0 \frac{\cos E_0 - e}{1 - e\cos E_0}$$

$$k_0 = -k_1 \cos E_0 - k_2 \sin E_0 - \overline{W}_0$$

and

$$C - g_0 = -\left[\left(1 - \frac{1}{2} e^2\right) \sin E_0 - \frac{e}{4} \sin 2E_0\right] k_1 + \left[\cos E_0 - \frac{e}{4} \cos 2E_0\right] k_2$$

$$- (n\delta z)_0 \quad .$$

33. Add the constants of integration to the series coefficients:

a. In $n\delta z$ add

$$\left(C - g_0\right) + \left(k_0 - \frac{1}{2} ek_1\right) nt + \left[\left(1 - \frac{e^2}{2}\right) k_1\right] \sin E$$

$$+ \left(-\frac{e}{4} k_1\right) \sin 2E + \left(-k_2\right) \cos E + \left(\frac{e}{4} k_2\right) \cos 2E \ ;$$

b. In ν add

$$\left(-\frac{2}{3} k_0 - \frac{e}{6} k_1\right) + \left(-\frac{1}{2} k_1\right) \cos E + \left(-\frac{1}{2} k_2\right) \sin E \ ;$$

c. In u add

$$\left(-l_1 e\right) + \left(l_1\right) \cos E + \left(l_2\right) \sin E \quad .$$

34. Print out the coefficients in the series for the perturbations.

Appendix B

Some First Order Perturbations

First order general perturbations due to Jupiter are given for six minor planets:

(13) Egeria
(1286) Banachiewizca
(132) Aethra
(241) Germania
(1274) Delportia
(1373) 1935 QN.

The orbital elements of the minor planets were taken from the Ephemerides of Minor Planets for 1962 (Reference B1). The elements of Jupiter were taken from Clemence (Reference B2) using the values given for 1950:

$$M' = 302^\circ\!.36489 + 0^\circ\!.08308578116\,(\mathrm{JD}-2433282.0)$$

$$\left.\begin{aligned} \omega' &= 274.14275 \\ \Omega' &= 99.80204 \\ i' &= 1.30710 \end{aligned}\right\} \quad \text{Ecliptic and Mean Equinox 1950.0} \qquad \begin{aligned} e' &= 0.04846063 \\ a' &= 5.20298098 \text{ a.u.} \\ n' &= 0^\circ\!.08308578116/\text{day} \end{aligned}$$

$$m' = 1/1047.355$$

The value of m' is taken from the American Ephemeris and Nautical Almanac.

(13) Egeria

Epoch 1938 Dec 8.0 ET = JD 2429240.5
T = .0001 (JD-2429240.5)

M_0 = 31°864
ω = 78.013
Ω = 43.563
i = 16.537
(ω, Ω, i: Ecliptic and Mean Equinox 1950.0)
M_0' = 326.57371

e = 0.086199424
a = 2.5770 a.u.
n 0°23825639 per day

A	B	C
− 1.2775339	− 2.1720939	+ 0.5054949
+ 1.6594002	− 1.3084250	− 1.4705535
+ 1.5017412	− 0.4020153	+ 2.0549637

Egeria is the planet for which Hansen originally computed general perturbations (Reference B3) and later published tables (Reference B4). Further investigations and comparisons with observations were made by Hoelling (Reference B5) and Samter (References B6 and B7).

PERTURBATIONS OF EGERIA

I	J	$(N\delta z \cdot 10^4)$ degrees COS	SIN	$\nu \cdot 10^6$ COS	SIN	$u \cdot 10^6$ COS	SIN
0	0	5363 T	0 T	8 T	0 T	-19 T	
1	0	-293 T	-105 T	92 T	-256 T	224 T	-1557 T
2	0	6 T	2 T				
0	0	363	0	-218	-0	22	0
1	0	-715	631	-545	-622	-106	-403
2	0	17	-20	9	2	-2	-1
-2	1	-1	1	1	0	0	0
-1	1	17	-11	-11	-19	14	11
0	1	-27	46	8	-8	28	52
1	1	-321	112	-68	-196	-24	-37
2	1	8	1	-3	2	-5	-24
3	1	-2	-0	-0	-3	-0	1
-1	2	1	-1	0	-3	1	-0
0	2	36	33	30	-31	-23	43
1	2	-462	-492	217	-190	58	-61
2	2	-379	-523	546	-396	17	-25
3	2	10	11	2	1	-5	3
4	2	0	1	-1	0	0	-0
-1	3	1	1	-0	-0	0	-0
0	3	-20	-50	-52	21	18	-33
1	3	-1860	-1658	-237	106	59	16
2	3	26	2029	-1728	20	-457	-196
3	3	-37	7	-63	-46	-4	-1
4	3	0	-1	-0	-1	1	1
0	4	-0	-1	-1	0	-0	-1
1	4	2	-7	-6	-0	6	2
2	4	-10	90	-56	-13	-14	-20
3	4	-48	15	-20	-53	1	-16
4	4	13	4	-6	17	0	1
1	5	2	-1	-2	-1	2	-0
2	5	7	53	-15	-2	-2	-2
3	5	-39	2	-4	-38	12	-15
4	5	7	8	-10	8	-3	1
5	5	-0	-4	5	0	0	-0
1	6	-1	1	1	1	-2	0
2	6	0	55	7	3	1	-2
3	6	41	-8	6	34	-14	10
4	6	1	5	-5	2	-3	-1
5	6	2	-3	3	2	1	1
6	6	-1	0	-0	-2	-0	-0
3	7	5	-1	0	3	-1	1
4	7	1	2	-3	1	-1	-1
5	7	1	-1	1	1	-0	1
6	7	-1	-0	0	-1	0	-0
3	8	5	-1	0	1	-0	0
4	8	1	2	-2	1	-1	-1
5	8	0	-0	0	1	-0	0
6	8	-0	-0	0	-1	0	-0
7	8	0	0	-1	0	-0	-0
3	9	1	-0	0	-0	-0	-0
4	9	-1	-2	1	-1	0	0

(1286) Banachiewizca

Epoch 1951 Dec 20.0 ET = JD 2434000.5
T = .0001 (JD-2434000.5)

$M_0 = 215^\circ\!.072$

$\left.\begin{array}{l}\omega = 100.709 \\ \Omega = 201.315 \\ i = 9.707\end{array}\right\}$ Ecliptic and Mean Equinox 1950.0

$e = 0.093256863$

$a = 3.0219$ a.u.

$n = 0^\circ\!.18762278$ per day

$M_0' = 2.06202$

A	B	C
+ 1.5869835	+ 2.5537900	− 0.1852088
− 2.5133799	+ 1.4943933	− 0.7496642
− 0.5443193	+ 0.5453495	+ 2.9215716

The general perturbations of Banachiewizca were previously computed and given by Herget (Reference B8) as an example.

PERTURBATIONS OF BANACHIEWIZCA

		$(N\delta z \cdot 10^4)$ degrees		$\nu \cdot 10^6$		$u \cdot 10^6$	
I	J	COS	SIN	COS	SIN	COS	SIN
0	0	-9587 T	0 T	-25 T	0 T	12 T	
1	0	-633 T	309 T	-271 T	-552 T	-127 T	-1733 T
2	0	15 T	-7 T				
0	0	-4814	0	447	-0	10	0
1	0	-3496	-1822	1616	-3058	110	-84
2	0	81	36	9	-1	-3	2
-2	1	1	0	0	-1	-0	-0
-1	1	-16	4	7	17	6	-20
0	1	88	52	31	1	-21	-102
1	1	856	-257	139	457	4	60
2	1	-1	4	1	17	18	47
3	1	2	-1	2	3	-1	-1
-1	2	-2	-1	2	0	-0	-1
0	2	68	83	105	-74	-156	21
1	2	-2884	-5998	1094	-605	92	-49
2	2	-2380	-3346	3172	-2250	144	-37
3	2	58	87	-4	-2	-10	8
4	2	1	1	-1	1	0	-0
0	3	-5	-1	-1	8	6	-6
1	3	-87	-34	-11	76	-11	-40
2	3	853	-103	94	577	90	93
3	3	131	-195	233	174	6	13
4	3	-5	5	2	-1	-4	-3
5	3	-0	1	-1	-1	0	0
0	4	-0	-0	0	0	0	-1
1	4	-0	11	13	5	-15	-9
2	4	361	-372	132	74	21	1
3	4	-162	-358	357	-150	46	-78
4	4	56	27	-24	70	-3	2
5	4	-1	-1	1	1	1	-2
1	5	-1	-1	-2	2	3	-0
2	5	2	-59	-21	11	0	-10
3	5	190	78	-52	146	23	43
4	5	34	-35	37	47	13	3
5	5	-1	13	-25	-0	-1	-1
6	5	0	-0	-1	1	1	-0
2	6	-1	1	1	2	-1	-3
3	6	66	-14	11	24	5	3
4	6	1	-48	50	4	12	-12
5	6	12	9	-11	16	0	4
6	6	-6	2	-3	-9	0	-1
2	7	-0	-1	-1	-0	1	1
3	7	56	-66	-12	-4	1	-3
4	7	46	64	-51	38	1	23
5	7	8	-4	2	11	3	2
6	7	-2	5	-7	-2	-1	0
7	7	-1	-2	3	-2	0	0
3	8	-0	-0	-0	0	0	-1
4	8	11	4	-1	6	1	1
5	8	4	-6	6	5	3	-1
6	8	2	3	-4	2	-0	1
7	8	-2	0	0	-3	-0	-0
8	8	1	-1	1	1	0	0
3	9	-1	1	1	1	-1	-3
4	9	1189	-314	14	20	-1	3
5	9	-7	-102	90	-5	16	-31
6	9	2	3	-1	2	1	1
7	9	-1	1	-1	-1	-1	0
8	9	-0	-1	1	-0	0	-0
9	9	0	0	-0	1	-0	0
5	10	1	2	-1	1	0	0
6	10	1	-1	0	1	1	0
7	10	0	1	-1	-0	-0	0
8	10	-1	-0	0	-1	-0	-0
9	10	0	-0	0	1	0	0
5	11	4	1	-0	1	-0	0
6	11	1	-2	2	1	1	-0
6	12	0	1	-0	0	-0	0

(132) Aethra

Epoch 1925 Jan 10.0 ET = JD 2424160.5
T = .0001 (JD-2424160.5)

$$M_0 = 145^\circ\!.191$$

$$\left.\begin{aligned} \omega &= 253.349 \\ \Omega &= 259.662 \\ i &= 25.161 \end{aligned}\right\}$$ Ecliptic and Mean Equinox 1950.0

$$M_0' = 264.49794$$

$e = 0.38276405$
$a = 2.6123$ a.u.
$n = 0^\circ\!.23344222$ per day

A	B	C
− 2.0941885	− 1.0306821	− 1.0926239
+ 1.4719146	− 1.8667986	− 0.7579083
− 0.5214907	− 1.1300793	+ 2.2485237

Aethra is an interesting planet having a large eccentricity and inclination. Its general perturbations were previously computed by Herget (Reference B9). Accurate special perturbations using Musen's method (Reference B10) were included in a differential correction by Musen (Reference B11) and revealed certain errors in the observations. There is a term of long period corresponding to $i = 5$, $j = 14$.

PERTURBATIONS OF AETHRA

I	J	$(n\delta z \cdot 10^4)$ degrees COS	SIN	$\nu \cdot 10^6$ COS	SIN	$u \cdot 10^6$ COS	SIN
0	0	16316 T	0 T	-353 T	0 T	182 T	
1	0	581 T	980 T	-923 T	507 T	-475 T	3504 T
2	0	-56 T	-101 T				
0	0	8984	0	-1230	-0	-156	0
1	0	-1097	2087	-1960	-967	268	572
2	0	112	-230	19	10	2	-1
3	0	-1	3	-2	-1	0	1
-3	1	0	-0	-0	-0	1	-0
-2	1	-1	10	6	1	1	3
-1	1	7	-57	-67	-6	-75	-48
0	1	-3	-59	-59	15	-136	-97
1	1	65	50	-37	64	177	134
2	1	33	-14	11	42	33	34
3	1	-10	3	-3	-8	-2	-1
4	1	1	-1	0	0	0	-1
-2	2	-3	1	0	2	-1	1
-1	2	-10	2	-14	-20	-4	-24
0	2	125	-110	-105	-276	-119	-287
1	2	-2237	282	-175	-999	83	191
2	2	-375	100	-119	-509	40	139
3	2	43	-17	5	-14	-1	-7
4	2	3	1	-1	3	0	-0
-2	3	-1	-0	-0	0	-1	-0
-1	3	52	10	1	-6	4	-4
0	3	-256	10	-72	433	-134	481
1	3	-9949	-4702	-760	1821	-57	198
2	3	2531	1037	-964	2349	231	-781
3	3	-230	-96	-16	37	6	-13
4	3	-1	2	-5	4	-0	1
5	3	-1	-1	1	-1	-0	0
-1	4	2	3	1	-1	1	1
0	4	-4	-3	-16	16	-22	12
1	4	-81	-104	-143	90	-70	58
2	4	260	460	-333	193	86	-70
3	4	23	38	-83	47	29	-19
4	4	-7	-10	3	-3	-2	1
5	4	0	-1	3	-0	-0	-0
0	5	0	6	-4	-0	-4	-2
1	5	-7	-55	-87	-6	-77	-3
2	5	-153	681	-292	-49	29	0
3	5	-23	198	-220	-29	91	1
4	5	4	-35	14	2	-5	-1
5	5	0	1	0	1	0	-0
6	5	-0	1	-1	-1	0	0
0	6	3	-3	-0	-1	0	-1
1	6	-11	14	27	25	25	30
2	6	-395	299	123	145	38	25
3	6	228	-213	194	207	-89	-68
4	6	-10	8	15	15	-7	-6
5	6	-3	3	-2	-2	1	1
6	6	-0	-0	-0	-1	-0	0
1	7	0	-0	0	3	-1	3
2	7	-22	3	0	28	3	17
3	7	92	3	1	69	-4	-18
4	7	20	-3	3	28	-3	-13
5	7	-6	1	-1	-4	0	2
6	7	1	-0	0	0	-0	-0
1	8	1	1	-0	0	-1	0
2	8	-10	-6	-12	14	-11	13
3	8	134	156	-46	52	2	1
4	8	53	35	-37	54	13	-30
5	8	-10	-6	2	-4	-1	2
6	8	1	1	-0	1	0	-0
2	9	-0	2	4	-0	4	0
3	9	1	37	21	-2	6	-3
4	9	-7	-39	35	-7	-13	6
5	9	-1	-1	5	-2	-3	1
6	9	1	2	-1	1	1	-0
3	10	-2	3	4	3	3	1
4	10	11	-12	10	7	-2	-1
5	10	2	-5	6	2	-4	-1
6	10	-1	2	-1	-0	1	0
3	11	-3	1	1	5	1	5
4	11	98	1	4	19	2	1
5	11	20	-9	8	20	-7	-9
6	11	-3	2	-1	-1	1	1
4	12	3	2	2	-2	0	-1
5	12	-5	-3	2	-4	-0	2
6	12	-1	0	0	-1	0	1
4	13	0	1	1	-0	0	-0
5	13	-0	-3	2	-1	-0	0
6	13	-1	-1	1	-1	-1	1
4	14	-1	2	3	2	3	2
5	14	252	-281	14	8	2	-1
6	14	5	-19	18	5	-10	1
7	14	-1	2	-0	-0	0	-0
6	15	-1	0	-0	-1	0	0

(241) Germania

Epoch 1951 Dec 20.0 ET = JD 2434000.5
T = .0001 (JD-2434000.5)

$$M_0 = 237^\circ\!.440$$

$$\left.\begin{aligned} \omega &= 74.116 \\ \Omega &= 271.529 \\ i &= 5.516 \end{aligned}\right\}\ \text{Ecliptic and Mean Equinox 1950.0}$$

$$M_0' = 2.06202$$

e = 0.095828379
a = 3.0524 a.u.
n = $0^\circ\!.18481972$ per day

A	B	C
+ 2.9435085	+ 0.7494468	− 0.2933036
− 0.8069138	+ 2.6685677	− 1.2160512
− 0.0423471	+ 1.2444521	+ 2.7843020

First order general perturbations of Germania were previously computed by Kline and Herget (Reference B12) and compared with special perturbations in order to draw conclusions concerning the accuracy of first order general perturbations.

		(nδz · 10⁴) degrees		$\nu \cdot 10^6$		$u \cdot 10^6$	
I	J	COS	SIN	COS	SIN	COS	SIN
0	0	18838 T	0 T	-18 T	0 T	-33 T	
1	0	-673 T	220 T	-193 T	-587 T	342 T	-1184 T
2	0	16 T	-5 T				
0	0	-1278	0	-1228	-0	24	0
1	0	-1094	5961	-5205	-960	-136	-77
2	0	28	-147	4	3	-2	0
-1	1	1	9	11	-2	-9	-6
0	1	64	-17	30	-6	-73	-18
1	1	496	-896	476	262	29	16
2	1	4	-6	26	15	27	27
3	1	1	0	1	2	-1	-1
-1	2	-1	2	-0	-1	-0	1
0	2	22	-62	-74	-64	55	82
1	2	-7361	3764	-703	-1251	-58	-49
2	2	-4265	2778	-2607	-4009	-38	-81
3	2	115	-73	3	8	3	10
4	2	0	0	-0	1	-0	-0
0	3	1	1	1	-1	-3	1
1	3	-17	42	35	-8	-2	19
2	3	-309	-316	198	-236	-30	-84
3	3	-277	29	-27	-334	1	-7
4	3	10	-0	-1	3	-1	4
1	4	-2	-2	-3	-1	6	2
2	4	-214	145	-56	-42	-7	-4
3	4	57	233	-226	44	-12	74
4	4	-61	-33	35	-80	2	-1
5	4	2	1	-1	0	-1	1
1	5	0	-0	-0	-0	-0	1
2	5	4	8	2	-6	2	2
3	5	-73	10	-11	-53	-14	-4
4	5	-7	27	-30	-13	-6	8
5	5	-13	-17	24	-19	1	-0
6	5	0	1	-0	-0	-1	0
3	6	-0	19	-7	2	-1	1
4	6	19	7	-9	19	5	5
5	6	-7	5	-6	-9	-3	1
6	6	-1	-8	12	-2	0	0
3	7	5	1	-1	-1	0	-0
4	7	-6	12	-10	-4	-3	2
5	7	2	2	-3	2	0	1
6	7	-3	1	-0	-4	-1	-0
7	7	1	-3	5	2	0	0
4	8	2	1	-0	1	0	0
5	8	2	-1	1	2	1	-0
6	8	0	1	-1	0	-0	1
7	8	-1	-0	1	-2	-0	-0
8	8	1	-1	2	2	-0	0
4	9	7	-7	-1	-0	-0	-0
5	9	2	5	-4	2	0	2
6	9	0	-0	-0	0	0	0
7	9	-0	0	-1	-0	-0	0
8	9	-0	-0	1	-0	-0	-0
9	9	1	-0	0	1	-0	0
5	11	-0	-1	0	-0	0	0
6	11	-1	-0	0	-0	-0	-0

(1274) Delportia

Epoch 1932 Oct 20.0 ET = JD 2427000.5
T = .0001 (JD-2427000.5)

M_0 = 214.°74119
ω = 242.98033
Ω = 327.42878   Ecliptic and Mean Equinox 1950.0
i = 4.40990
M_0' = 140.46156

e = 0.1130022
a = 2.2290079 a.u.
n = 0.°29616788 per day

A	B	C
− 1.9192056	+ 1.1226357	− 0.0922681
− 0.9698111	− 1.7192724	− 1.0167650
− 0.5870198	− 0.8299533	+ 1.9814519

First order general perturbations of Delportia were previously computed by Herget (Reference B13) and used to produce the ephemeris presently in use.

PERTURBATIONS OF DELPORTIA

I	J	$(n\delta z \cdot 10^4)$ degrees COS	SIN	$\nu \cdot 10^6$ COS	SIN	$u \cdot 10^6$ COS	SIN
0	0	-6491 T	0 T	-5 T	0 T	13 T	
1	0	-648 T	48 T	-42 T	-565 T	-113 T	431 T
2	0	18 T	-1 T				
0	0	95	0	106	-0	-8	0
1	0	570	256	-214	497	36	-36
2	0	-16	-8	1	0	1	0
-1	1	-1	2	3	1	-7	3
0	1	12	79	3	2	-11	6
1	1	71	251	-166	47	14	-3
2	1	-2	-4	-2	0	5	-0
-1	2	-0	0	0	0	-1	1
0	2	0	8	-9	8	-8	8
1	2	204	341	-186	113	15	-16
2	2	135	209	-235	152	5	-2
3	2	-5	-7	0	0	-1	0
0	3	-1	2	-4	4	-5	6
1	3	284	388	-102	80	3	-3
2	3	248	280	-269	237	13	-24
3	3	-27	-26	23	-26	-1	0
4	3	1	1	0	-0	0	-0
0	4	0	-0	1	-1	1	-2
1	4	74	81	24	-24	1	-3
2	4	-157	-134	110	-129	-6	18
3	4	-14	-8	14	-22	-1	2
4	4	5	2	-3	7	0	-0
1	5	1	1	1	-2	0	-1
2	5	-23	-16	10	-16	-0	2
3	5	-12	-5	6	-13	-0	2
4	5	5	1	-2	6	0	-0
5	5	-1	-0	0	-2	-0	0
2	6	-10	-5	2	-5	-0	0
3	6	-9	-3	3	-9	0	1
4	6	3	0	-0	3	-0	-0
5	6	-1	-0	-0	-2	0	0
6	6	0	-0	0	1	0	-0
2	7	-102	-39	2	-7	-0	-0
3	7	-29	-5	4	-26	1	4
4	7	2	0	-0	1	-0	-0
5	7	-1	0	-0	-1	0	0
6	7	0	-0	0	1	-0	-0
3	8	2	0	-0	1	-0	-0
4	8	1	-0	0	1	-0	-0

(1373) 1935 QN

Epoch 1941 Jan 6.0 ET = JD 2430000.5
T = .0001 (JD-2430000.5)

$M_0 = 293\overset{\circ}{.}612$

$\left.\begin{array}{l} \omega = 99.051 \\ \Omega = 298.068 \\ i = 38.902 \end{array}\right\}$ Ecliptic and Mean Equinox 1950.0

$M_0' = 29.71890$

$e = 0.32158820$

$a = 3.4111$ a.u.

$n = 0\overset{\circ}{.}15644444$ per day

A	B	C
+ 2.0607314	− 1.8497219	− 1.8902002
+ 0.7243470	+ 2.5384346	− 1.9809114
+ 2.6199827	+ 0.7530860	+ 2.0343883

(1373) was chosen as an example because it is one of the most interesting minor planets. It has a large eccentricity and inclination and is the only known planet for which there exists a libration in the argument of perihelion (Reference B14). The secular perturbations have been computed by Smith (Reference B15) using Halphen's method. Smith's results verify the libration in perigee and disclose large secular perturbations in the eccentricity and inclination. There is no dominant small divisor for small values of the indices, and the series for the perturbations converges very slowly. The term $i = 8$, $j = 15$ will contain some inaccuracy since the series are not computed beyond $j = 15$. There will be another significant term at $i = 9$, $j = 17$. A further study of this planet will be given in the future.

PERTURBATIONS OF (1373)

I	J	($n\delta z \cdot 10^4$) degrees COS	SIN	$\nu \cdot 10^6$ COS	SIN	$u \cdot 10^6$ COS	SIN
0	0	10279 T	0 T	292 T	0 T	80 T	
1	0	3220 T	-987 T	908 T	2810 T	-249 T	-3505 T
2	0	-259 T	84 T				
0	0	40369	0	-925	-0	183	0
1	0	-2563	1577	-1496	-2220	-334	-1959
2	0	197	-178	64	-14	16	-9
3	0	-2	2	3	-5	-7	3
4	0	1	2	-3	2	0	1
5	0	0	-0	-0	1	1	-0
-4	1	0	0	-0	-1	1	0
-3	1	1	-4	-6	-1	1	-1
-2	1	-6	-3	4	6	-17	-0
-1	1	20	71	111	-24	14	23
0	1	69	292	115	0	-191	85
1	1	342	545	-284	208	149	-78
2	1	100	120	-154	125	75	-35
3	1	-24	-33	27	-19	3	-6
4	1	-2	0	2	-5	-2	2
5	1	1	1	-1	1	-0	1
6	1	0	-0	-0	1	0	-0
-4	2	-0	0	1	0	-0	0
-3	2	1	0	0	-1	1	0
-2	2	-0	-4	-7	1	3	-3
-1	2	-107	-127	-6	14	-19	1
0	2	761	1108	1385	-1073	-941	998
1	2	31997	28101	3784	-3847	-88	47
2	2	-3985	-4468	4120	-3720	668	-673
3	2	402	416	-38	66	6	-4
4	2	-12	-8	8	-10	1	-4
5	2	-2	0	1	-4	-1	1
6	2	1	0	-0	1	-0	1
-3	3	0	0	1	-0	-0	0
-2	3	1	1	1	-2	2	0
-1	3	-2	-6	-8	4	4	-7
0	3	5	15	4	27	-38	9
1	3	-161	-20	-22	229	77	-218
2	3	1494	335	-212	786	-91	208
3	3	137	115	-146	245	-14	30
4	3	5	-8	-6	35	1	2
5	3	-5	-2	2	-4	0	-2
6	3	-1	0	-0	-3	-0	0
-2	4	0	0	1	-0	-0	1
-1	4	2	0	0	-3	2	0
0	4	-0	-12	-8	7	4	-12
1	4	-50	49	78	83	-145	33
2	4	511	2041	368	290	44	-210
3	4	1460	-98	115	1206	-26	413
4	4	-85	24	-15	40	-1	-0
5	4	8	-3	3	16	1	2
6	4	-2	-0	1	-1	-0	-1
7	4	-1	0	-1	-1	-0	-0
-1	5	0	0	1	-0	-0	1
0	5	3	-0	-0	-3	2	0
1	5	-11	1	-7	13	0	-21
2	5	-16	-26	-47	31	52	2
3	5	121	394	-185	-23	-39	-26
4	5	-165	50	-73	-168	-14	-55
5	5	19	-2	-4	8	-2	-2
6	5	4	-3	4	6	1	1
7	5	-0	-0	1	-0	-0	-0
8	5	-0	0	-1	-1	-0	-0

PERTURBATIONS OF (1373) CONTINUED

I	J	(nδz · 10⁴) degrees COS	SIN	$\nu \cdot 10^6$ COS	SIN	$u \cdot 10^6$ COS	SIN
0	6	0	0	1	-0	-0	1
1	6	6	-2	-2	-4	3	0
2	6	-31	4	-1	43	-15	-65
3	6	-568	158	-14	155	45	29
4	6	21	247	-198	21	-83	-56
5	6	-25	-1	-23	-27	-4	-8
6	6	3	1	-3	2	-1	-1
7	6	1	-2	2	2	0	0
8	6	-0	-0	1	0	0	-0
1	7	0	1	1	-0	0	1
2	7	-1	-2	-4	-4	6	0
3	7	17	-17	-25	-30	14	36
4	7	-319	186	-37	-110	-7	-24
5	7	-1	-68	60	-16	32	12
6	7	-3	11	-8	-6	-1	-2
7	7	1	1	-2	1	-1	-0
8	7	0	-1	1	1	0	0
2	8	1	1	1	-1	0	2
3	8	1	-9	-14	-4	17	-2
4	8	-16	-119	-54	-13	0	14
5	8	-90	62	-44	-68	6	-20
6	8	3	-15	11	-5	6	1
7	8	-1	3	-3	-1	-0	-1
8	8	0	1	-1	1	-0	0
2	9	-0	0	0	0	-1	0
3	9	1	-1	1	-1	1	2
4	9	6	9	15	-9	-19	4
5	9	-79	-205	62	-11	14	-2
6	9	29	-15	24	26	-5	12
7	9	-4	-1	3	-3	2	-0
8	9	-0	1	-1	-0	0	-0
3	10	-0	0	0	0	-1	0
4	10	2	-1	1	-3	2	4
5	10	26	-8	4	-15	-8	1
6	10	-23	-43	30	-15	10	0
7	10	5	0	4	3	-1	2
8	10	-1	-1	1	-1	1	-0
4	11	1	0	1	0	-1	0
5	11	-3	3	5	6	-3	-8
6	11	116	-72	6	28	2	5
7	11	14	23	-20	17	-9	-0
8	11	-0	-3	1	1	-0	1
9	11	-0	-0	0	-1	0	-0
5	12	0	1	1	-0	-2	1
6	12	3	8	5	0	-0	-3
7	12	14	-9	5	10	-0	4
8	12	1	2	-2	2	-1	-0
6	13	-2	-2	-4	4	5	-4
7	13	104	118	-18	8	-3	3
8	13	-16	9	-11	-13	1	-7
9	13	2	-1	-0	1	-0	-0
6	14	-0	0	-0	0	-0	-1
7	14	-2	1	-0	2	1	-1
8	14	5	6	-4	3	-1	0
9	14	-1	1	-1	-1	-0	-1
7	15	2	-2	-2	-4	1	9
8	15	-374	242	-5	-19	-3	-5
9	15	-10	-18	15	-11	8	6
10	15	1	2	-0	-0	0	-0

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